GRAPH THEORY FINAL EXAM

This exam is of 50 marks. There are 8 questions. You may use any theorem you have learned as long as you state it correctly and carefully. Please do not cheat. Good luck! (50)

Notation

- $v = v(G)$ - no. of vertices. $e = e(G)$ - no. of edges. $w = w(G)$ - no. of components.
- $d(v)$ - degree of a vertex. $\delta = \delta(G)$ - smallest degree. $\Delta = \Delta(G)$ - largest degree.
- $N(S)$ - the set of vertices adjacent to a set of vertices S.
- χ' - edge chromatic number.
- χ - vertex chromatic number.
- G is k-critical if $\chi = k$ and for all $H \subset G$ proper subgraphs, $chi(H) < \chi(G)$
- α - independence number - no of vertices in a maximum independent set.
- $r(k, l)$ - Ramsey number.
- $\pi_k(G)$ - Chromatic polynomial of G.

1a. Show that if G is a bipartite graph, then it has a perfect matching if and only if (4)

$$|N(S)| \geqslant |S|$$

for all $S \subset V$. Here $N(S)$ is the set of vertices adjacent to the vertices in S.

1b. Give an example to show it does not hold if the bipartite assumption is dropped. (4)

2. Show that if G is a non-empty regular simple graph then $\chi' = \Delta + 1$. (4)

3. A connected graph is α-critical if $\alpha(G - e) > \alpha(G)$ for all $e \in E$. Show that an α-critical graph has no cut vertices. (4)

4. Show that $r(k, l) = r(l, k)$, where $r(k, l)$ is the Ramsey number. (4)

5. In a group of 9 people, one person knows two others, two people each know 4 others, four each know 5 others and the remaining two know six others. Show that there are three people who know each other. (6)

6. Show that G is 3-critical if and only if G is an odd cycle. (4)

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7. Let G be a simple graph on ν vertices, with e edges and c components.

a. Show that the coefficient of $k^{\nu-1}$ in $\pi_k(G)$ is $-e$. (6)

b. Show that the $\pi_k(G)$ has a zero of order at least c at $k = 0$. (4)

8a. Recall that a g-toroidal graph is one which can be embedded in a g-holed torus such that all of its faces are simply connected. Suppose G is a connected g-toroidal graph with ν vertices, e edges and f faces, show that (6)

$$\nu - e + f = 2 - 2g$$

8b. Show that $K_{3,3}$ is not planar. (4)